# VEKTORANALYS

Kursvecka 4

# CURVILINEAR COORDINATES
# (kroklinjiga koordinatsytem)

Kapitel 10
Sidor 99-121

# TARGET PROBLEM

- An athlete is rotating a hammer
- Calculate the force on the arms.

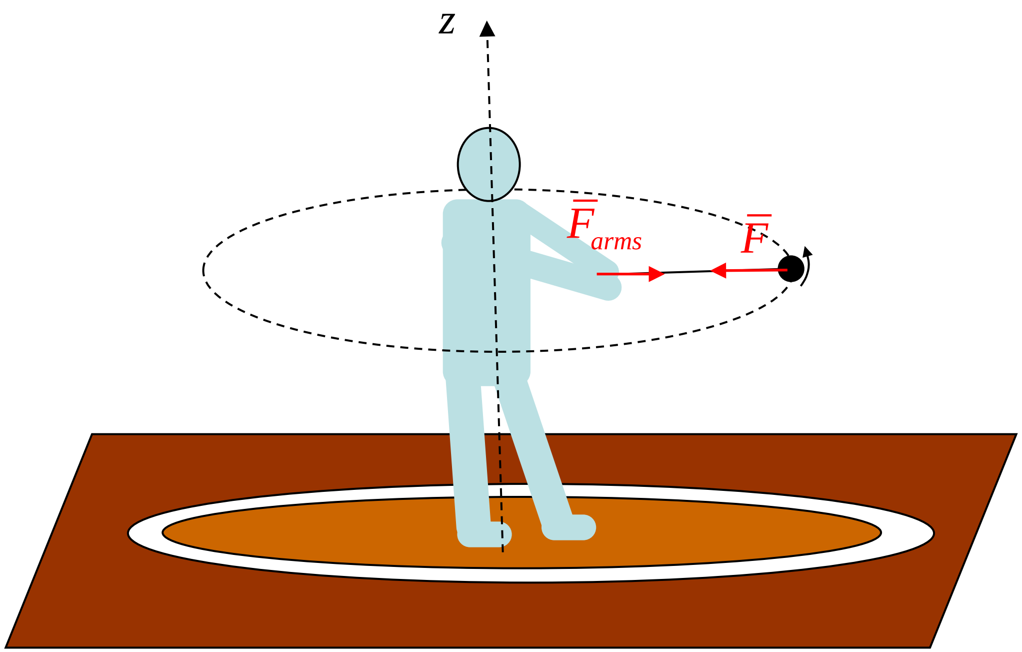
$$\bar{F}_{arms} = -\bar{F}$$

$$\bar{F} = m\bar{a}$$

$$\bar{a} = \frac{d\bar{v}}{dt} \equiv \dot{\bar{v}}$$

$$\bar{v} = \frac{d\bar{r}}{dt} \equiv \dot{\bar{r}}$$

$$\bar{r} = \rho_0 \hat{e}_\rho + z_0 \hat{e}_z$$ *(in cylindrical coord.)*




We need

- to introduce **curvilinear coordinates**
- to describe **cylindrical coordinates**
- to calculate **the derivative of** $\hat{e}_\rho$

# CYLINDRICAL COORDINATES

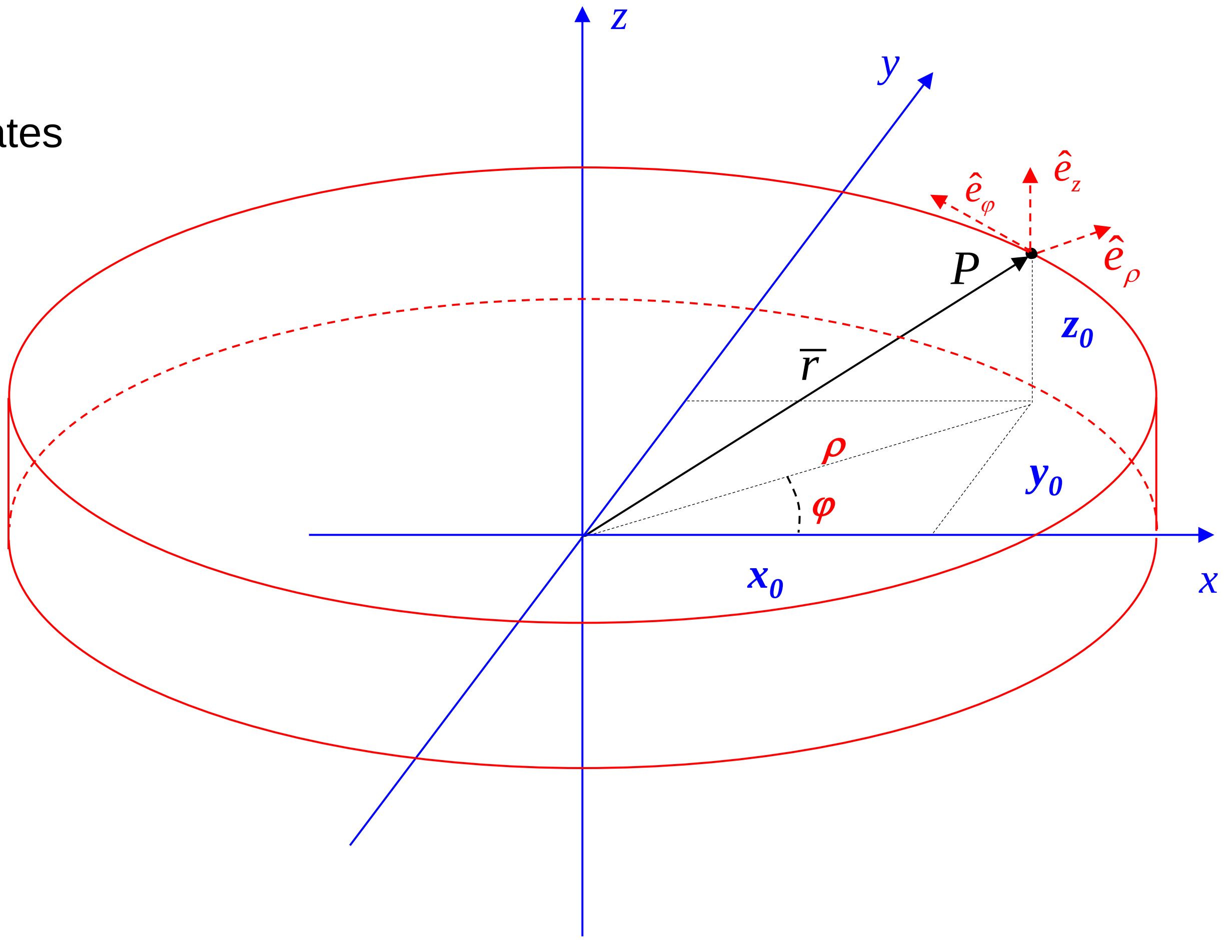
Cylindrical coordinates are an example of curvilinear coordinates

*cartesian coord.*

$P:\ x_0,\ y_0,\ z_0$

*cylindrical coord.*

$P:\ \rho,\ \varphi,\ z_0$




$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \tan\varphi = y/x \\ z = z \end{cases}$$

$$0 \le \rho \le \infty$$

$$0 \le \varphi \le 2\pi$$

$$-\infty \le z \le +\infty$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

# CYLINDRICAL COORDINATE SYSTEM: "informal introduction"

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

$$\bar{r} = (x, y, z) = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\bar{r} = (\rho\cos\varphi, \rho\sin\varphi, z) =$$

$$= \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\hat{e}_z$$

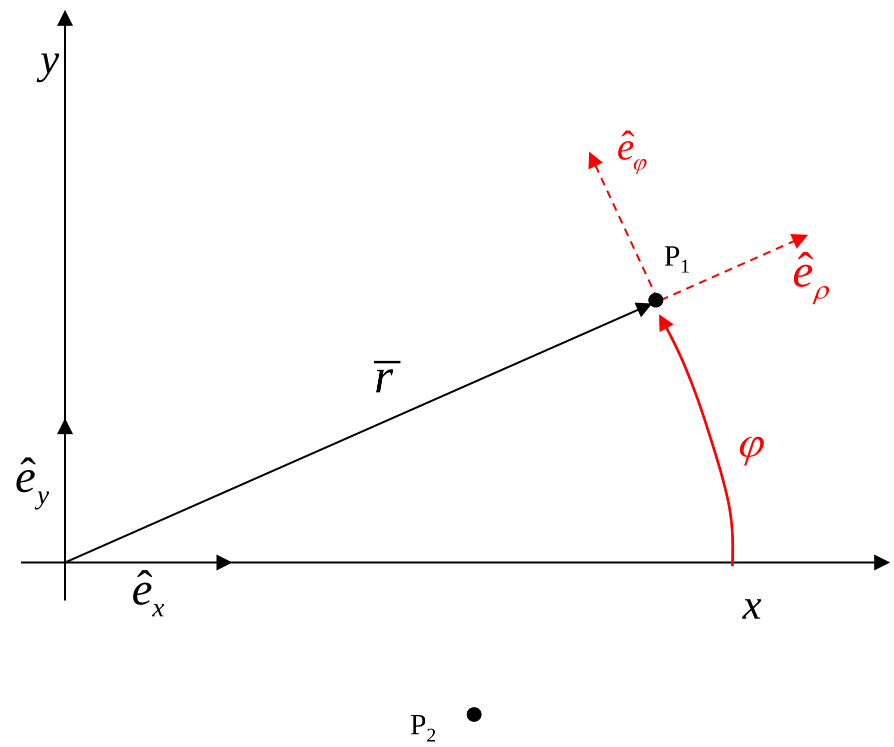
**(it is still in a cartesian coordinate system)**

$$\frac{\partial\bar{r}}{\partial x} = \frac{\partial(x,y,z)}{\partial x} = (1,0,0) = \bar{e}_x$$

*This is the rate of variation of $\bar{r}$ with x keeping y and z constant ⇒ it is a vector parallel to the x-axis*

$$\frac{\partial\bar{r}}{\partial y} = \frac{\partial(x,y,z)}{\partial y} = (0,1,0) = \bar{e}_y$$

*This is the rate of variation of $\bar{r}$ with y keeping x and z constant ⇒ it is a vector parallel to the x-axis*




EXERCISE:
Plot $\bar{e}_\rho$ and $\bar{e}_\varphi$ in P2

How to calculate in a cylindrical coordinate system the $\bar{e}_\rho$ axis and the $\bar{e}_\varphi$ axis?

$$\frac{\partial\bar{r}}{\partial\rho} = \frac{\partial(\rho\cos\varphi, \rho\sin\varphi, z)}{\partial\rho} = (\cos\varphi, \sin\varphi, 0) = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y = \bar{e}_\rho$$

*This is the rate of variation of $\bar{r}$ with $\rho$ keeping $\varphi$ and z constant ⇒ it is a vector parallel to the $\rho$-axis*

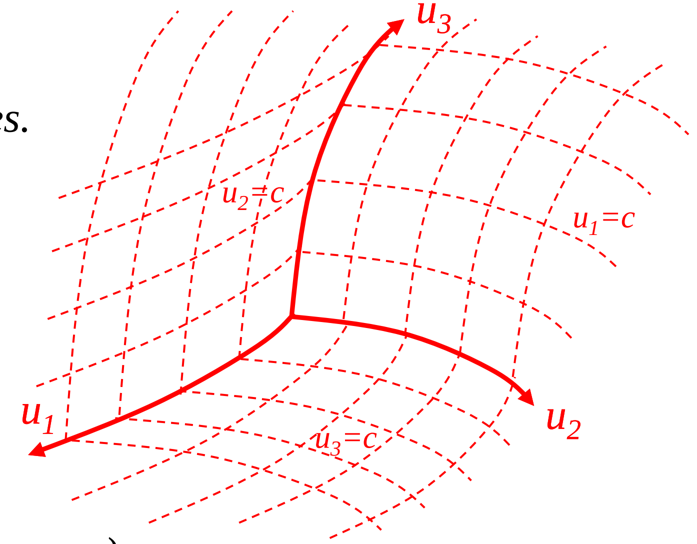
$$\frac{\partial\bar{r}}{\partial\varphi} = \frac{\partial(\rho\cos\varphi, \rho\sin\varphi, z)}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0) = -\rho\sin\varphi\hat{e}_x + \rho\cos\varphi\hat{e}_y = \bar{e}_\varphi$$

*This is the rate of variation of $\bar{r}$ with $\varphi$ keeping $\rho$ and z constant ⇒ it is a vector parallel to the $\varphi$-axis*

EXERCISE:
Express in a cylindrical coordinate system (i.e. using $\hat{e}_\rho$ and $\hat{e}_\varphi$) the vector:

$\bar{v} = x\hat{e}_x + y\hat{e}_y$

# CURVILINEAR COORDINATE SYSTEMS




Consider a cartesian coordinate system *x, y, z*
*The surfaces x=c  y=c and z=c are the coordinate surfaces.*
*Their intersection defines the x-axis, y-axis and z-axis*

Consider another coordinate system defined by the variables $u_1, u_2, u_3$

We assume that there is a one-to-one relationship between $x_i$ and $u_i$,
so that $x_i$ can be expressed as a function of $u_i$ *(and vice-versa)*:

$$\begin{cases} x = x(u_1, u_2, u_3) \\ y = y(u_1, u_2, u_3) \\ z = z(u_1, u_2, u_3) \end{cases}$$

- The surfaces defined by $u_i=c$ are called coordinate surfaces
- The curves defined by the intersection of the coordinate surfaces are called coordinate curves
- The 3 curves $u_1, u_2, u_3$ are the coordinate axes

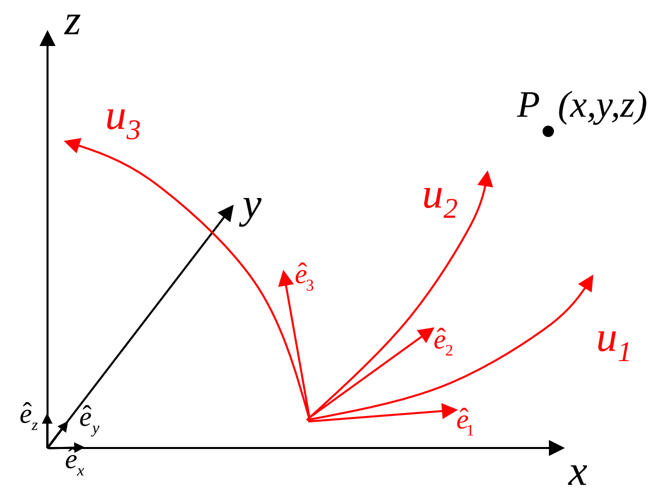
DEFINITION

A system of curvilinear coordinates is orthogonal if the coordinate curves are perpendicular to each other where they intersect

$$\overline{u}_i \cdot \overline{u}_j = \delta_{ik}$$

# CURVILINEAR COORDINATES

| | Point P | the basis is defined by the unit vectors: | $d\bar{r}$ |
|---|---|---|---|
| CARTESIAN | $x, y, z$ | $\hat{e}_x \quad \hat{e}_y \quad \hat{e}_z$ | $(dx, dy, dz)$ |
| CURVILINEAR | $u_1, u_2, u_3$ | ? | ? |




DEFINITION

An orthogonal curvilinear coordinate system has an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ in each point and

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \quad with \quad scale\ factor \quad h_i = \left|\frac{\partial \bar{r}}{\partial u_i}\right|$$

Orthonormal:

- magnitude 1 $\quad |\hat{e}_i| = \left|\frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i}\right| = \frac{1}{|h_i|}\left|\frac{\partial \bar{r}}{\partial u_i}\right| = 1$
- orthogonal $\quad \hat{e}_i \cdot \hat{e}_j = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial u_i} \cdot \frac{1}{h_j}\frac{\partial \bar{r}}{\partial u_j} = 0 \quad for\ i \neq j$

$$\Rightarrow \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

Kronecker delta

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u_1}du_1 + \frac{\partial \bar{r}}{\partial u_2}du_2 + \frac{\partial \bar{r}}{\partial u_3}du_3 = h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$$

$$d\bar{r} = \sum_i h_i du_i \hat{e}_i$$

# SURFACE ELEMENT AND VOLUME ELEMENT

In a Cartesian coordinate system:

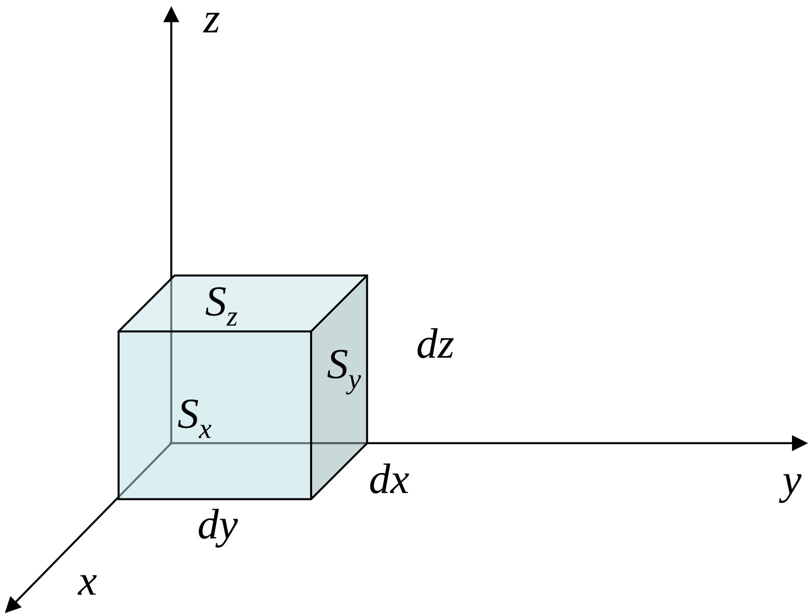
$$d\bar{r} = (dx, dy, dz) = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z$$

$$dS_x = dydz$$

$$dS_y = dxdz$$

$$dS_z = dxdy$$

$$dV = dxdydz$$




In a orthogonal curvilinear coordinate system:

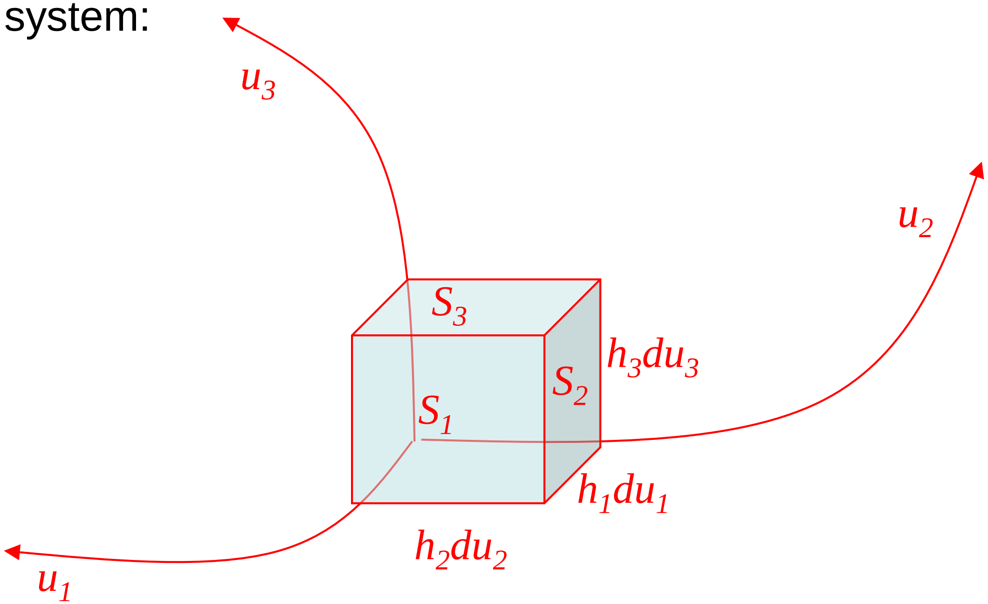
$$d\bar{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$dS_1 = h_2 h_3 du_2 du_3$$

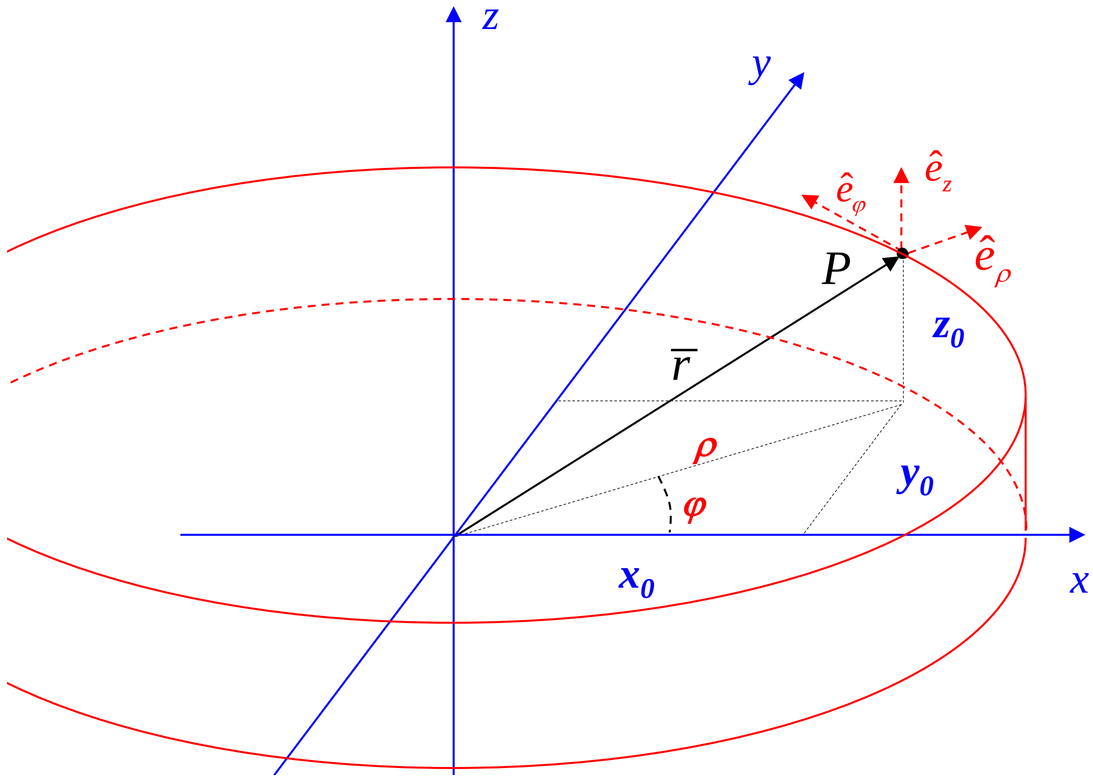
$$dS_2 = h_1 h_3 du_1 du_3$$

$$dS_3 = h_1 h_2 du_1 du_2$$

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$



# CYLINDRICAL COORDINATES




## Orthonormal basis

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

$$\overline{r} = \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$$

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial\overline{r}}{\partial u_i} \quad with \quad h_i = \left|\frac{\partial\overline{r}}{\partial u_i}\right|$$

$$\begin{cases} \dfrac{\partial\overline{r}}{\partial\rho} = (\cos\varphi, \sin\varphi, 0) \\ \dfrac{\partial\overline{r}}{\partial\varphi} = (-\rho\sin\varphi, \rho\cos\varphi, 0) \\ \dfrac{\partial\overline{r}}{\partial z} = (0,0,1) \end{cases}$$

$$\begin{cases} h_\rho = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1 \\ h_\varphi = \sqrt{(-\rho\sin\varphi)^2 + (\rho\cos\varphi)^2} = \rho \\ h_z = 1 \end{cases}$$

$$\begin{cases} \hat{e}_\rho = \dfrac{1}{h_\rho}\dfrac{\partial\overline{r}}{\partial\rho} = (\cos\varphi, \sin\varphi, 0) = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \hat{e}_\varphi = \dfrac{1}{h_\varphi}\dfrac{\partial\overline{r}}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0) = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \\ \hat{e}_z = \dfrac{1}{h_z}\dfrac{\partial\overline{r}}{\partial z} = (0,0,1) = \hat{e}_z \end{cases}$$

$$dS_\rho = \rho\, d\varphi\, dz$$
$$dS_z = \rho\, d\varphi\, d\rho$$
$$dV = \rho\, d\rho\, d\varphi\, dz$$

EXERCISE: *write* $\overline{r}$ *using* $\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z$

# TARGET PROBLEM

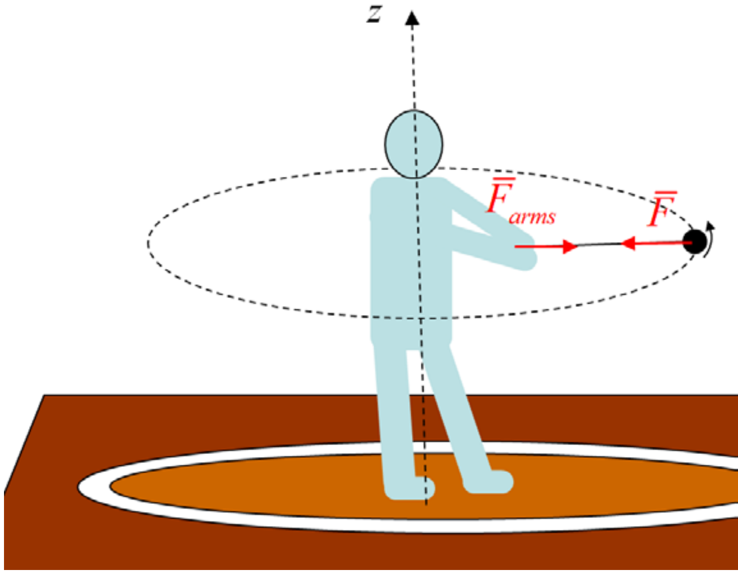
$$\bar{F}_{arms} = -\bar{F}$$

$$\bar{F} = m\bar{a}$$

$$\bar{a} = \frac{d\bar{v}}{dt} \equiv \dot{\bar{v}}$$

$$\bar{v} = \frac{d\bar{r}}{dt} \equiv \dot{\bar{r}}$$

$$\bar{r} = \rho_0 \hat{e}_\rho + z_0 \hat{e}_z$$ *(in cylindrical coord.)*




$$\bar{F}_{arms} = -m\bar{a} = -m\dot{\bar{v}} = -m\ddot{\bar{r}}$$

$$\ddot{\bar{r}} = \frac{d}{dt}\dot{\bar{r}} = \frac{d}{dt}\left(\dot{\rho}_0 \hat{e}_\rho + \rho_0 \dot{\hat{e}}_\rho + \dot{z}_0 \hat{e}_z + z_0 \dot{\hat{e}}_z\right) =$$

$\dot{\rho}_0$ =0 (the length does not change)

$\dot{z}_0$ =0 (the hammer rotates on a plane z=constant)

$$\dot{\hat{e}}_\rho = \frac{d}{dt}(\cos\varphi, \sin\varphi, 0) = (-\dot{\varphi}\sin\varphi, \dot{\varphi}\cos\varphi, 0) = \dot{\varphi}(-\sin\varphi, \cos\varphi, 0) = \dot{\varphi}\hat{e}_\varphi$$

$$\dot{\hat{e}}_z = \frac{d}{dt}(0,0,1) = 0$$

$$= \frac{d}{dt}\left(\rho_0 \dot{\varphi}\hat{e}_\varphi\right) = \left(\dot{\rho}_0 \dot{\varphi}\hat{e}_\varphi + \rho_0 \ddot{\varphi}\hat{e}_\varphi + \rho_0 \dot{\varphi}\dot{\hat{e}}_\varphi\right) = \left(\dot{\rho}_0 \dot{\varphi}\hat{e}_\varphi + \rho_0 \ddot{\varphi}\hat{e}_\varphi - \rho_0 \dot{\varphi}\dot{\varphi}\hat{e}_\rho\right) = \rho_0\left(\ddot{\varphi}\hat{e}_\varphi - \dot{\varphi}^2\hat{e}_\rho\right)$$

$$\dot{\hat{e}}_\varphi = \frac{d}{dt}(-\sin\varphi, \cos\varphi, 0) = (-\dot{\varphi}\cos\varphi, -\dot{\varphi}\sin\varphi, 0) = -\dot{\varphi}(\cos\varphi, \sin\varphi, 0) = -\dot{\varphi}\hat{e}_\rho$$

$$\bar{F}_{arms} = -m\ddot{\bar{r}} = m\rho_0\left(\dot{\varphi}^2\hat{e}_\rho - \ddot{\varphi}\hat{e}_\varphi\right)$$

# PRACTICAL EXAMPLE: **THE BIOT-SAVART LAW**

The magnetic field in a point P of a steady line current is given by the Biot-Savart law:

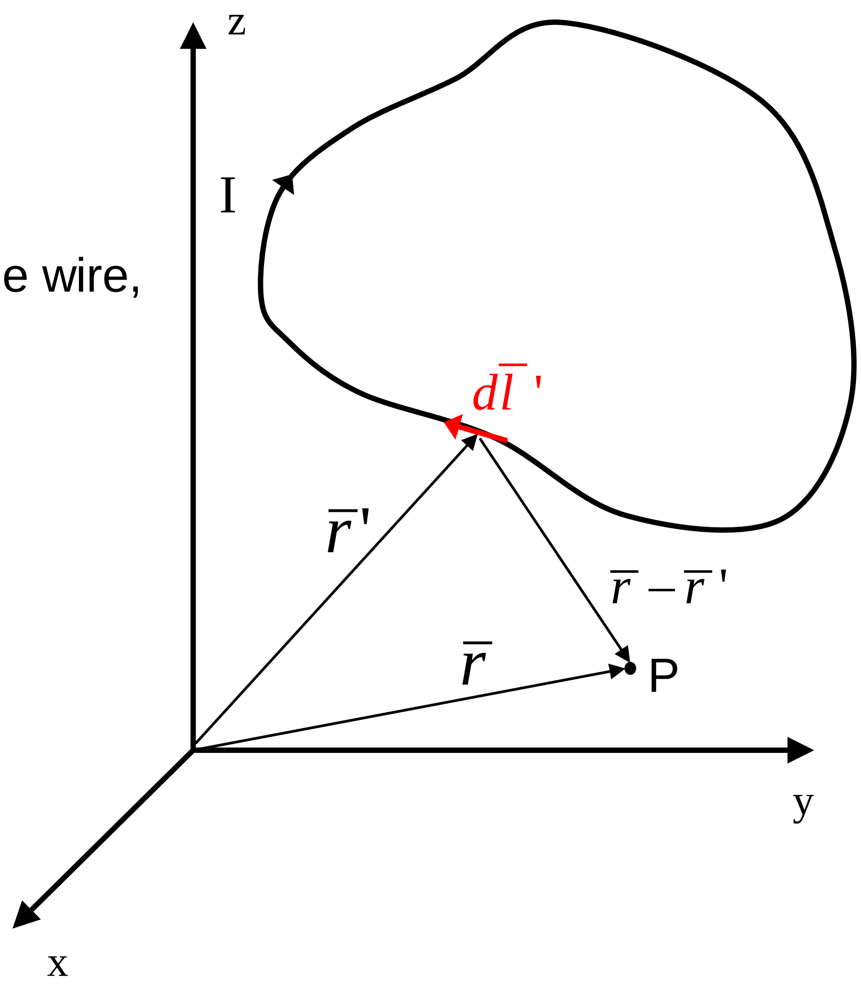
$$\bar{B}(\bar{\mathrm{r}}) = \frac{\mu_0 I}{4\pi} \oint_L \frac{d\bar{l}\,' \times (\bar{r} - \bar{r}\,')}{\left|\bar{r} - \bar{r}\,'\right|^3}$$

Where $d\bar{l}\,'$ is an infinitesimal length along the wire,

$\bar{r}$ is the position vector of the point P and

$\bar{r}\,'$ is a vector from the origin to $d\bar{l}\,'$

Therefore, $\bar{r} - \bar{r}\,'$ is a vector from $d\bar{l}\,'$ to P

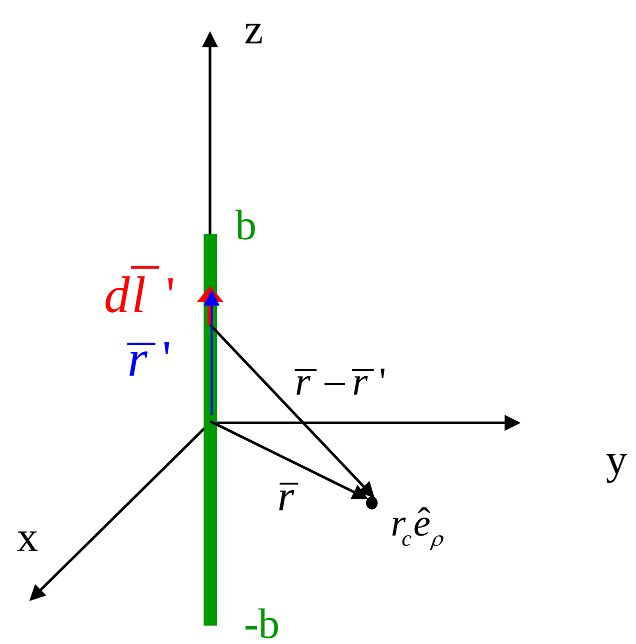


# PRACTICAL EXAMPLE: **THE BIOT-SAVART LAW**

The magnetic field in a point P of a steady line current is given by the Biot-Savart law:

$$\bar{B}(\bar{r}) = \frac{\mu_0 I}{4\pi}\int_L \frac{d\bar{l}\,'\times(\bar{r}-\bar{r}\,')}{|\bar{r}-\bar{r}\,'|^3}$$

Calculate the magnetic field in P= $x_0\hat{e}_x + y_0\hat{e}_y$ produced by a straight wire along the z-axis with current I and length 2b and centred at z=0.




SOLUTION:

*If* $r_c = \sqrt{x_0^2 + y_0^2}$ *is the distance from the origin to P,*

*in cylindrical coordinates:* $\bar{r} = r_c\hat{e}_\rho$

$$\bar{l}(z') = (0,0,z') = z'\hat{e}_z \quad \Rightarrow d\bar{l}\,' = dz'\hat{e}_z$$

*with* $z' : -b \rightarrow +b$

$$\bar{r}\,' = z'\hat{e}_z$$

$$\bar{r}-\bar{r}\,' = r_c\hat{e}_\rho - z'\hat{e}_z \quad \Rightarrow \quad |\bar{r}-\bar{r}\,'| = \sqrt{r_c^2 + z'^2}$$

$$d\bar{l}\,'\times(\bar{r}-\bar{r}\,') = dz'\hat{e}_z \times (r_c\hat{e}_\rho - z'\hat{e}_z) = r_c dz'\hat{e}_\varphi$$

$$\bar{B}(\bar{r}) = \frac{\mu_0 I}{4\pi}\int_{-b}^{b}\frac{r_c dz'\hat{e}_\varphi}{(r_c^2+z'^2)^{3/2}} = \hat{e}_\varphi \frac{\mu_0 I}{4\pi} r_c \int_{-b}^{b}\frac{dz'}{(r_c^2+z'^2)^{3/2}} = \hat{e}_\varphi \frac{\mu_0 I}{4\pi} r_c \left[\frac{z'}{r_c^2\sqrt{r_c^2+z'^2}}\right]_{-b}^{b} = \frac{\mu_0 I}{4\pi r_c}\frac{2b}{\sqrt{r_c^2+b^2}}\hat{e}_\varphi$$

*If the wire is infinitely long, we need to calculate the limit for* $b\rightarrow\infty \Rightarrow \quad \bar{B}(\bar{r}) = \frac{\mu_0 I}{2\pi r_c}\hat{e}_\varphi$

# WHICH STATEMENT IS WRONG?

1- $\hat{e}_\rho$ does not depend on the position (yellow)

2- $\hat{e}_z$ is constant everywhere (red)

3- Cylindrical coordinates are appropriate with a cylindrical symmetry (green)

4- Cylindrical coordinates are a curvilinear coordinate system (blue)

# TARGET PROBLEM

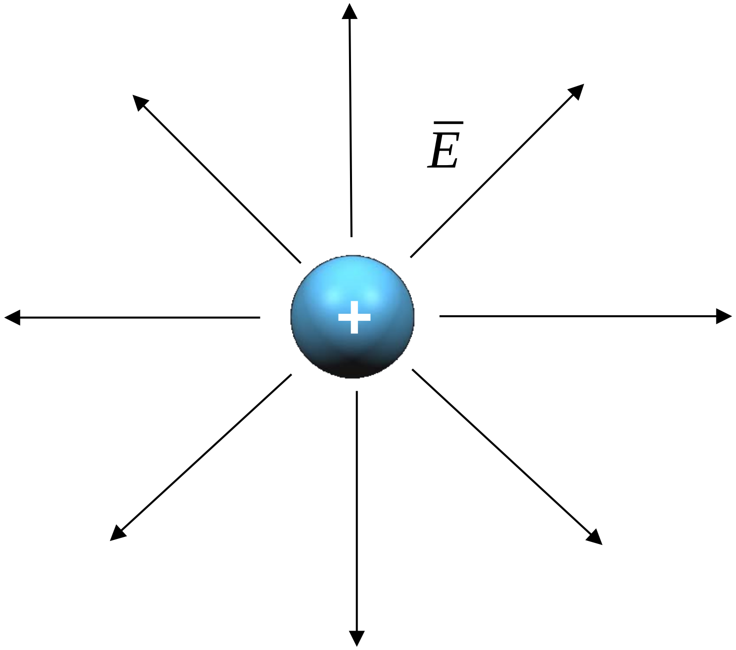
The electric field is conservative. The electrostatic potential $\phi$ is defined by: $\bar{E} = -\nabla\phi$

$$\left.\begin{aligned} \bar{E} &= -\nabla\phi \\ \nabla\cdot\bar{E} &= 0 \end{aligned}\right\} \Longrightarrow \nabla\cdot(\nabla\phi) = \boxed{\nabla^2\phi = 0}$$

*first Maxwell's equation with no charge*

Laplace's equation

Calculate the electrostatic potential generated outside a spherical charge.




Due to the spherical symmetry, the solution will depend only on the radius: $\phi = \phi(r)$

with $r = |\bar{r}|$

We need to:

- introduce spherical coordinates
- calculate gradient and divergence in spherical coordinates
- solve the equation

# SPHERICAL COORDINATES

spherical coordinates are an example of curvilinear coordinates

*cartesian coord.*

$P$: $x_0$, $y_0$, $z_0$

*spherical coord.*

$P$: $r$, $\theta$, $\varphi$

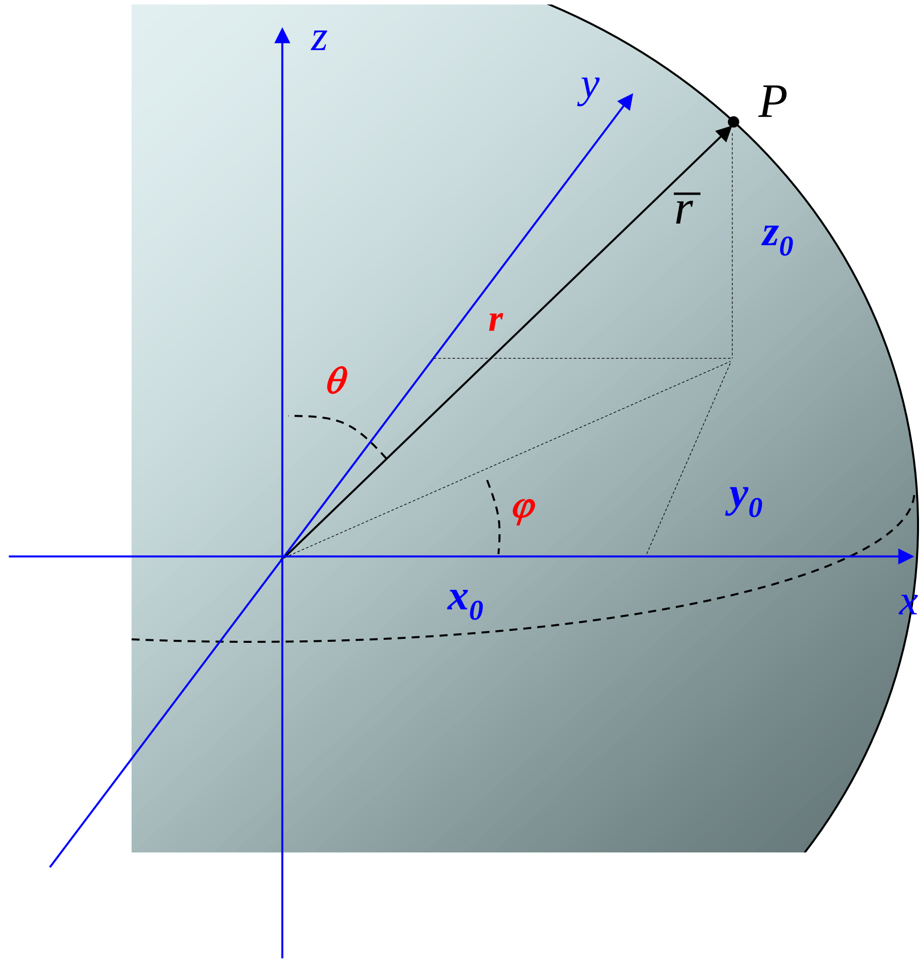
$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \tan\theta = \dfrac{\sqrt{x^2 + y^2}}{z} \\ \tan\varphi = y / x \end{cases}$$

$$0 \le r \le \infty$$

$$0 \le \theta \le \pi$$

$$0 \le \varphi \le 2\pi$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



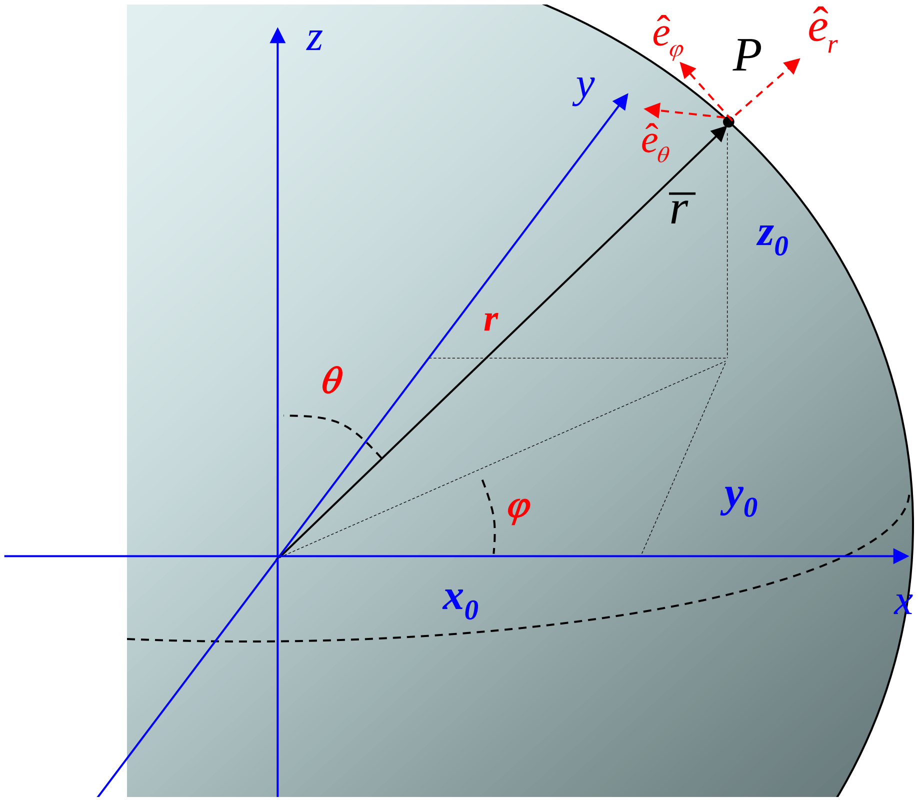
# SPHERICAL COORDINATES

## Orthonormal basis

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\bar{r} = r\sin\theta\cos\varphi\,\hat{e}_x + r\sin\theta\sin\varphi\,\hat{e}_y + r\cos\theta\,\hat{e}_z$$

$$\hat{e}_i = \frac{1}{h_i}\frac{\partial\bar{r}}{\partial u_i} \quad with \quad h_i = \left|\frac{\partial\bar{r}}{\partial u_i}\right|$$




$$\begin{cases} \dfrac{\partial\bar{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \dfrac{\partial\bar{r}}{\partial\theta} = (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta) \\ \dfrac{\partial\bar{r}}{\partial\varphi} = (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0) \end{cases}$$

$$\begin{cases} h_r = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = 1 \\ h_\theta = \sqrt{(r\cos\theta\cos\varphi)^2 + (r\cos\theta\sin\varphi)^2 + (r\sin\theta)^2} = r \\ h_\varphi = \sqrt{(r\sin\theta\sin\varphi)^2 + (r\sin\theta\cos\varphi)^2} = r\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \hat{e}_r = \dfrac{1}{h_r}\dfrac{\partial\bar{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \hat{e}_\theta = \dfrac{1}{h_\theta}\dfrac{\partial\bar{r}}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\ \hat{e}_\varphi = \dfrac{1}{h_\varphi}\dfrac{\partial\bar{r}}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0) \end{cases}$$

$$dS_r = r^2\sin\theta\, d\theta\, d\varphi$$
$$dV = r^2\sin\theta\, dr\, d\theta\, d\varphi$$

EXERCISE: *write* $\bar{r}$ *using* $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$

# GRADIENT IN CURVILINEAR COORDINATES

In cartesian coordinates: $$grad\phi = \left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right) = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$$

And in a curvilinear coordinate system?
We must express $grad\phi$ in terms of the curvilinear basis $\hat{e}_1,\hat{e}_2,\hat{e}_3$:

$$grad\phi = g_1\hat{e}_1 + g_2\hat{e}_2 + g_3\hat{e}_3$$

since $d\phi = grad\phi\cdot d\bar{r}$ and $d\bar{r} = h_1du_1\hat{e}_1 + h_2du_2\hat{e}_2 + h_3du_3\hat{e}_3$

$$d\phi = \left(g_1\hat{e}_1 + g_2\hat{e}_2 + g_3\hat{e}_3\right)\cdot\left(h_1du_1\hat{e}_1 + h_2du_2\hat{e}_2 + h_3du_3\hat{e}_3\right) = g_1h_1du_1 + g_2h_2du_2 + g_3h_3du_3$$

But also, writing $\phi$ as a function of $u_i$: $$d\phi = \frac{\partial\phi}{\partial u_1}du_1 + \frac{\partial\phi}{\partial u_2}du_2 + \frac{\partial\phi}{\partial u_3}du_3$$

Therefore: $$g_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1},\quad g_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial u_2},\quad g_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}$$

$$\boxed{grad\phi = \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}\hat{e}_i}$$

# GRADIENT IN CURVILINEAR COORDINATES

THE GRADIENT

- in cylindrical coordinates:

$$grad\phi = \frac{1}{h_\rho}\frac{\partial \phi}{\partial \rho}\hat{e}_\rho + \frac{1}{h_\varphi}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi + \frac{1}{h_z}\frac{\partial \phi}{\partial z}\hat{e}_z = \boxed{\frac{\partial \phi}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi + \frac{\partial \phi}{\partial z}\hat{e}_z}$$

- in spherical coordinates:

$$grad\phi = \frac{1}{h_r}\frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{h_\theta}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{1}{h_\varphi}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi = \boxed{\frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\hat{e}_\varphi}$$

EXERCISE: *calculate in spherical coordinates* $\nabla\left(\frac{1}{r}\right)$

DIVERGENCE IN CURVILINEAR COORD.

$$div\bar{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(A_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(A_2 h_3 h_1\right) + \frac{\partial}{\partial u_3}\left(A_3 h_1 h_2\right)\right]$$

*Proof: see theorem 10.4, page 109*

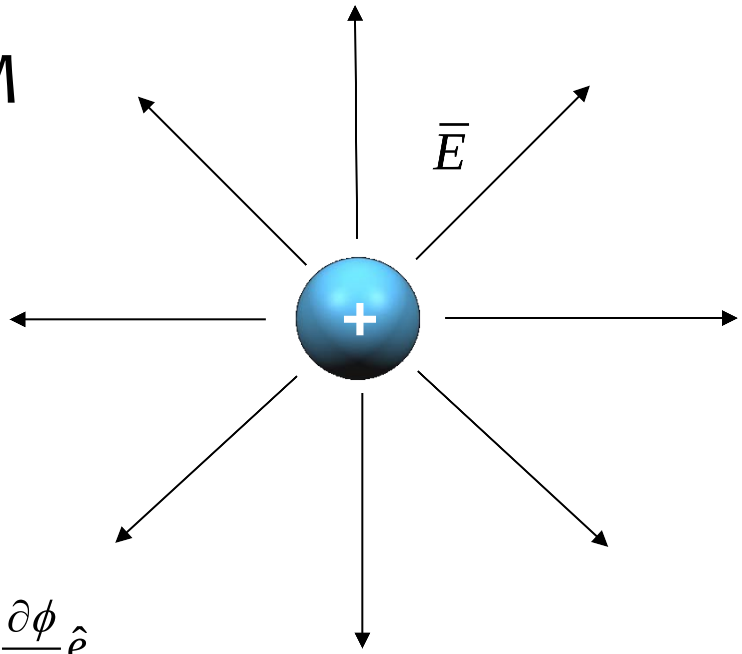
EXERCISE: *calculate in spherical coordinates* $\nabla \cdot \bar{A}$

CURL IN CURVILINEAR COORD.

$$rot\bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

*Proof: see theorem 10.5, page 112*

# TARGET PROBLEM




$$\nabla^2\phi = 0$$

*Which can be written as:*

$$\nabla^2\phi = \nabla\cdot\left(\nabla\phi\right) = 0$$

*Due to spherical symmetry*

$$\phi = \phi(r)$$

$$\text{with } \; r = \left|\bar{r}\right|$$

*Due to spherical symmetry, the solution is easy in spherical coordinates*

$$grad\,\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi$$

$$div\bar{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial u_2}\left(A_2h_3h_1\right) + \frac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right] = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(A_r r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(A_\theta r\sin\theta\right) + \frac{\partial}{\partial\varphi}\left(rA_\varphi\right)\right]$$

$$div\left(grad\,\phi\right) = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r}r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\underbrace{\frac{\partial\phi}{\partial\theta}}_{=0}r\sin\theta\right) + \frac{\partial}{\partial\varphi}\left(r\frac{1}{r\sin\theta}\underbrace{\frac{\partial\phi}{\partial\varphi}}_{=0}\right)\right] = \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2}$$

$\phi = \phi(r)$ $\Rightarrow$ *No* $\theta$ *and no* $\varphi$ *dependence*

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = 0 \quad \Rightarrow \quad \phi(r) = -\frac{c}{r} + d$$

$$\bar{E} = -grad\,\phi = \frac{c}{r^2}\hat{e}_r$$

# WHICH STATEMENT IS WRONG?

1- The scale factor is necessary to calculate the gradient (yellow)

2- The scale factor is necessary to calculate the divergence (red)

3- Spherical coordinates are a curvilinear coordinate system (blue)

4- In spherical coordinates the position vector is $\overline{r} = (r, \theta, \varphi)$ (green)